ON ALMOST p-REGULAR CONJUGACY CLASSES

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Abstract. We show that if G has a trivial maximal normal solvable subgroup then the number of almost p-regular conjugacy classes of G is greater than $2\sqrt{p-1}$.

1. Introduction

Let G be a finite group and p a prime which divides the order of G. The number $k(G)$ of conjugacy classes of G is a fundamental invariant in group and representation theory. Bounding $k(G)$ in terms of a prime divisor of $|G|$ is a fundamental problem related to [1, Problem 21]. A conjecture of Héthelyi and Külshammer [4] was that for any p-block B of any finite group G, the number of $k(B)$ of complex irreducible characters in B is 1 or is at least $2\sqrt{p-1}$. Analogously, $k(G) \geq 2\sqrt{p-1}$ for all G and p.

Proving $k(G) \geq 2\sqrt{p-1}$ for all G and p turned out to be a difficult problem. Building on a series of relevant works by Héthelyi-Külshammer [4, 5], Malle [9], Keller [8], and Héthelyi-Horváth-Keller-Maróti [3], the conjecture was confirmed in [10]. Further work [11] showed that there is some constant $c > 0$ such that for when the square of the prime divides the order of the group the bound $k(G) \geq cp$ holds.

We define the *p-regularity level* of an element $g \in G$ to be $\log_p(|g|_p)$, where $|g|_p$ is the p-part of the order of g. Here almost p-regular means p-regularity level 0 or 1. A series of recent works [11, 7, 6] has lead to the following conjecture.

Conjecture 1.1. Let G be a finite group of order divisible by p, for p a prime. Then,

$$k_{ap'}(G) \geq 2\sqrt{p-1}.$$

In this paper, we prove a stronger (strictly greater) bound for a special case of Conjecture 1.1.

Theorem 1.2. *Let G be a finite group of order divisible by p, for p a prime. If the maximal normal solvable subgroup of G is trivial, then*

$$k_{ap'}(G) > 2\sqrt{p-1}.$$

2. Main Result

In this section we prove Theorem 1.2. We first extend a result from [6] which is essential in our proof. When a group G acts on a set X, we use $n(G, X)$ to denote the number of G-orbits on X.

Theorem 2.1 (Hung, Maróti 2021)**.** *Let S be a non-abelian finite simple group and let p be a prime divisor of $|S|$. We have*

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Table 1. Possible exceptions for the bound $n(\mathrm{Aut}(S), \mathrm{Cl}_{p'}(S)) > 2\sqrt{p-1}$.

S	p	$n(\mathrm{Aut}(S), \mathrm{Cl}_{p'}(S))$
A_5	5	3
$\mathrm{PSL}_2(7)$	7	4
A_6	5	4
$\mathrm{PSL}_2(8)$	7	4
$\mathrm{PSL}_2(11)$	11	6
$\mathrm{PSL}_2(16)$	17	5
$\mathrm{PSL}_2(27)$	13	5
$\mathrm{PSL}_2(32)$	11	6
$\mathrm{PSL}_2(32)$	31	6
$\mathrm{PSL}_2(81)$	41	10
$\mathrm{PSL}_2(128)$	43	12
$\mathrm{PSL}_2(128)$	127	12
$\mathrm{PSL}_2(243)$	61	15
$\mathrm{PSL}_2(256)$	257	21
$\mathrm{PSL}_3(8)$	73	13
$\mathrm{PSU}_3(16)$	241	30
${}^2B_2(8)$	13	6
${}^2B_2(32)$	31	8
${}^2B_2(32)$	41	9
${}^2B_2(128)$	113	≥ 19
${}^2B_2(128)$	127	≥ 14
$\Omega_8^-(4)$	257	≥ 32

(i) *The number of* $\mathrm{Aut}(S)$*-orbits on the set* $\mathrm{Cl}_{p'}(S) \cup \mathrm{Cl}_p(S)$ *is larger than* $2\sqrt{p-1}$ *except if* (S,p) *is equal to* $(\mathsf{A}_5, 5)$ *or to* $(\mathrm{PSL}_2(16), 17)$.

(ii) *The number of* $\mathrm{Aut}(S)$*-orbits on* p*-regular classes of* S *is at least* $2(p-1)^{1/4}$. *The equality occurs if and only if* $(S,p) = (\mathrm{PSL}_2(16), 17)$.

(iii) *The number of* $\mathrm{Aut}(S)$*-orbits on* p*-regular classes of* S *is greater than* $2\sqrt{p-1}$ *unless possibly when* (S,p) *is listed in Table 1.*

We now extend Theorem 2.1 (iii) to almost p-regular classes, beginning with an obvious fact.

Lemma 2.2. *Let* G *be a finite group of order divisible by* p*, for* p *prime. Then,*

$$k_{ap'}(G) > k_{p'}(G).$$

Proof. Observe that $k_{ap'}(G) = k_{\mathrm{pr}-1}(G) + k_{\mathrm{pr}-0}(G) = k_{\mathrm{pr}-1}(G) + k_{p'}(G)$, by Cauchy's theorem we have that there exists at least one element g of order p, which would reside in the class $\mathrm{Cl}_{\mathrm{pr}-1}(G)$. Then, $k_{ap'}(G) \geq k_{p'}(G) + 1$ and the desired inequality follows. □

Lemma 2.3. *Let* S *be a non-abelian finite simple group of order divisible by* p*, for* p *prime. We have*

(i) $k_{ap'}(S) > 2\sqrt{p-1}$ *for* (S,p) *listed in Table 1.*

(ii) *The number of* $\mathrm{Aut}(S)$*-orbits on almost* p*-regular classes of* S *is greater than* $2\sqrt{p-1}$ *unless when* (S,p) *is listed in Table 2.*

TABLE 2. Exceptions for the bound $n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S)) > 2\sqrt{p-1}$.

S	p	$n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S))$
A_5	5	4
$\mathrm{PSL}_2(16)$	17	7
$\mathrm{PSL}_2(32)$	31	9
$\mathrm{PSL}_2(128)$	127	21
${}^2B_2(32)$	41	11

Proof. (i) For (S, p) listed in Table 1, we observe that $\mathrm{Cl}_{ap'}(S) = \mathrm{Cl}(S)$. From [10], we know that every finite group whose order is divisible by a prime p has at least $2\sqrt{p-1}$ conjugacy classes, thus $k_{ap'}(S) > 2\sqrt{p-1}$.

(ii) From Theorem 2.1 (iii) we know that S has more than $2\sqrt{p-1}$ orbits on conjugacy classes of p-regular elements of S unless possibly when (S, p) is as listed in Table 1. It follows then from Lemma 2.2 that S has more than $2\sqrt{p-1}$ orbits on conjugacy classes of almost p-regular elements of S unless possibly when (S, p) is as listed in Table 1. In that case, we compute $n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S))$ directly in GAP [2], which leaves us with possible exceptions left as in Table 2. For $(S, p) \in \{({}^2B_2(32), 31), ({}^2B_2(32), 41), ({}^2B_2(128), 113), ({}^2B_2(128), 127), (\Omega_8^-(4), 257)\}$, computing $n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S))$ directly in GAP is expensive, making either a possible exception. Observe that $\mathrm{Cl}_{ap'}(S) = \mathrm{Cl}(S)$, We then theoretically bound $n(\mathrm{Aut}(S), \mathrm{Cl}(S))$ by classifying group elements by their orders and computing the number of $\mathrm{Aut}(S)$-orbits under each conjugacy class.

Let $S = {}^2B_2(32)$, We directly compute the number of $\mathrm{Aut}(S)$-orbits under each conjugacy class in GAP. The trivial element of S forms 1 $\mathrm{Aut}(S)$-orbit. There is 1 class of elements of order 2 accounting for 1 $\mathrm{Aut}(S)$-orbit. There are 2 classes of elements of order 4 accounting for 2 $\mathrm{Aut}(S)$-orbits. There is 1 class of elements of order 5 accounting for 1 $\mathrm{Aut}(S)$-orbit. There are 5 classes of elements of order 25 accounting for 1 $\mathrm{Aut}(S)$-orbit. There are 15 classes of elements of order 31 accounting for 3 $\mathrm{Aut}(S)$-orbits. There are 10 classes of elements of order 41 accounting for 2 $\mathrm{Aut}(S)$-orbits.Then,

$$n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S)) = n(\mathrm{Aut}(S), \mathrm{Cl}(S)) = 11 > 2\sqrt{31-1}.$$

So we have that $({}^2B_2(32), 41)$ is an exception left as in Table 2.

Let $S = {}^2B_2(128)$, Suzuki [12] proved that S has $2^7 + 3$ conjugacy classes and $|\mathrm{Out}(S)| = 7$. The trivial element of S forms 1 $\mathrm{Aut}(S)$-orbit. There is 1 class of elements of order 2 accounting for 1 $\mathrm{Aut}(S)$-orbit. There are 2 classes of elements of order 4 accounting for 2 $\mathrm{Aut}(S)$-orbits. There are 28 classes of elements of order 113 forming at least 4 $\mathrm{Aut}(S)$-orbits. There are 63 classes of elements of order 127 forming at least 9 $\mathrm{Aut}(S)$-orbits. There are 36 classes of elements in a cyclic torus of order 145; 1 for elements of order 5, 7 for elements of order 29, and 28 for elements of order 145. Together these form at least 6 $\mathrm{Aut}(S)$-orbits. Then,

$$n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S)) = n(\mathrm{Aut}(S), \mathrm{Cl}(S)) \geq 1 + 1 + 2 + 4 + 9 + 6 = 23.$$

For both $p = 113$ and $p = 127$, $n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S)) > 2\sqrt{p-1}$. So we have that neither $({}^2B_2(128), 113)$ or $({}^2B_2(128), 127)$ are exceptions to the desired bound.

Table 3. Aut(S)-Orbits on Conjugacy Classes by element order.

$element'order$	$k(S)$	$\mathrm{Aut}(S) - orbits$
1	1	1
2	3	≥ 1
3	3	≥ 1
4	4	≥ 1
5	9	≥ 1
6	6	≥ 1
7	1	1
8	2	≥ 1
9	1	1
10	15	≥ 2
12	3	≥ 1
13	2	≥ 1
15	35	≥ 3
17	4	≥ 1
20	6	≥ 1
21	2	≥ 1
30	24	≥ 2
34	12	≥ 1
35	4	≥ 1
39	4	≥ 1
45	4	≥ 1
51	12	≥ 1
60	8	≥ 1
63	6	≥ 1
65	8	≥ 1
85	32	≥ 3
102	8	≥ 1
105	8	≥ 1
170	16	≥ 2
195	16	≥ 2
255	16	≥ 2
257	64	≥ 6
315	24	≥ 2

Let $S = \Omega_8^-(4)$, we know that S has 363 conjugacy classes computed in GAP and $|\operatorname{Out}(S)| = 12$. Classify the conjugacy classes based on the element orders, the number of $\mathrm{Aut}(S)$-orbits under each conjugacy class is presented in the following Table 3. Then,

$$n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S)) = n(\mathrm{Aut}(S), \mathrm{Cl}(S)) \geq 48 > 2\sqrt{p-1}$$

for $p = 257$.

□

Table 4. Exception for the group orders and prime factorizations

S	p	$\|S\|$	$\|\mathrm{Out}(S)\|$
A_5	5	$2^2 \cdot 3 \cdot 5$	2
$\mathrm{PSL}_2(16)$	17	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
$\mathrm{PSL}_2(32)$	31	$2^5 \cdot 3 \cdot 11 \cdot 31$	5
$\mathrm{PSL}_2(128)$	127	$2^7 \cdot 3 \cdot 43 \cdot 127$	7
${}^2B_2(32)$	41	$2^{10} \cdot 5^2 \cdot 31 \cdot 41$	5

We can now prove Theorem 1.2, which is restated below.

Theorem 2.4. *Let G be a finite group of order divisible by p, for p prime. If the maximal normal solvable subgroup of G is trivial, then*

$$k_{ap'}(G) > 2\sqrt{p-1}.$$

Proof. Since G has a trivial maximal normal subgroup, we have that the minimal normal subgroup N of G is non-abelian. Since N is non-abelian, we have that it is semi-simple and can be constructed as a direct product of k copies of a non-abelian simple group S.

We first consider when $k > 1$, that is $N \cong S \times \cdots \times S$. If p divides the order of S, then let n be the number of $\mathrm{Aut}(S)$-orbits on p-regular classes of S. Observe that,

$$k_{ap'}(G) > k_{p'}(G) \geq n(\mathrm{Aut}(N), \mathrm{Cl}_{p'}(N)) \geq \binom{n+k-1}{k} \geq \frac{n(n+1)}{2}.$$

Since $\mathrm{Aut}(N) \cong S^k \wr S_k$, where S^k is the direct product of S k-times and S_k is the symmetric group on k-elements. From Theorem 2.1 (ii) we know that $n \geq 2(p-1)^{1/4}$ and thus $k_{p'}(G) > 2\sqrt{p-1}$. Then, by Lemma 2.2 we have that $k_{ap'}(G) > 2\sqrt{p-1}$ and the case when $k > 1$ is handled.

Now consider when $k = 1$, that is $N = S \trianglelefteq G$. If p divides the order of S observe that,

$$k_{ap'}(G) \geq n(G, \mathrm{Cl}_{ap'}(S)) \geq n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S))$$

since the action of conjugation is an automorphism so each G-orbit is contained in an $\mathrm{Aut}(S)$-orbit. Particularly, the size of the $\mathrm{Aut}(S)$-orbits is larger than the size of the G-orbits and thus $n(G, \mathrm{Cl}_{ap'}(S)) \geq n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S))$. Thus, it suffices to show $n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S)) > 2\sqrt{p-1}$. From Lemma 2.3 we know that S has more than $2\sqrt{p-1}$ orbits on conjugacy classes of almost p-regular elements of S, unless when (S,p) is as listed in Table 2. It follows that if (S,p) is not one of the seven listed in Table 2, then we have that $k_{ap'}(G) > 2\sqrt{p-1}$ as desired.

To finish the proof, we must show that $k_{ap'}(G) > 2\sqrt{p-1}$ for all (S,p) listed in Table 2. Let (S,p) be one case in Table 2. As $N = S$ is a minimal normal subgroup of G, there are two cases to consider, either S is the unique minimal normal subgroup of G or there is at least some other minimal normal subgroup T such that $G = S \times T$.

If N is the unique minimal normal subgroup of G, we know that $p^2 \nmid |S|$ and $p^2 \nmid |\mathrm{Out}(S)|$ from Table 4. Then, we have $\mathrm{Cl}_{ap'}(G) = \mathrm{Cl}(G) > 2\sqrt{p-1}$ for $G \leq \mathrm{Aut}(S)$ from [10].

If N is not the unique minimal normal subgroup of G, there exists at least one other minimal normal subgroup of G such that $S \times T \leq G$, where $S \neq T$ otherwise this falls into the previous case of N having $k > 1$ copies of S in it. To compute $k_{ap'}(G)$ for all possible G would be very expensive so to handle this we first bound $n(\mathrm{Aut}(G), \mathrm{Cl}_{ap'}(G))$ by computing

$n(\mathrm{Aut}(G), \mathrm{Cl}_{ap'}(G)) = n(\mathrm{Aut}(S), \mathrm{Cl}_{ap'}(S)) \times n(\mathrm{Aut}(T), \mathrm{Cl}_{ap'}(T))$ directly. Since $G \cong S \times T$, and $G/S \times T \cong \mathrm{Out}(S \times T)$, thus $\mathrm{Aut}(S \times T) \cong \mathrm{Aut}(S) \times \mathrm{Aut}(T)$. From Lemma 2.3, it follows that the desired bound holds whether p divides only the order of the minimal normal subgroup S or T, or p divides the orders of both S and T. With this we have handled all possible exceptions in Table 2 and thus the proof is complete. □

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